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ANALOGUES OF THE KIRCHHOFF AND SOMIGLIANA FORMULAE IN TWO-DIMENSIONAL ELASTODYNAMIC PROBLEMS*

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The theory of generalized functions is used to derive unsteady-state equations of motion in elasticity theory taking into account possible discontinuities at the fronts of the solutions in infinite domains, and also for solutions in a bounded domain. By convolving Green's tensor with the right-hand side of these equations one obtains generalized-function analogues of the Kirchhoff, Somigliana and Gauss formulae. Integral analogues of these formulae are proposed for the case of two-dimensional deformation.

1. Unsteady-state equations of motion in generalized functions. We shall use the following notation: (x_1, x_2, x_3) are Lagrangian Cartesian coordinates of a point **x** in a linearly elastic isotropic medium with given Lamé parameters λ, μ and density ρ and $u_i, \varepsilon_{ij}, \sigma_{ij}$ are the Cartesian components of the displacements **u** and strain and stress tensors, respectively. These quantities obey the Cauchy relations and Hooke's law /1/:

$$\varepsilon_{ij} = 0.5\,(u_{i,j} + u_{j,i}), \quad \sigma_{ij} = \lambda u_{k,k}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{1.1}$$

Throughout the paper, repeated indices indicate summation; unless otherwise stated, $i, j = 1, \ldots, N$ (in two-dimensional deformation $N = 2$ and in three-dimensional $N = 3$) and $u_{i,j} = \partial u_i/\partial x_j,\ u_{i,t} = \partial u_i/\partial t$.

In view of (1.1), the equations of motion of a continuous medium

$$\sigma_{ij,j} + \rho G_i = \rho u_{i,tt} \tag{1.2}$$

can be reduced to the form

$$L_i^j\,(\partial/\partial\mathbf{x}, \partial/\partial t)\,u_j + G_i = 0 \tag{1.3}$$

$$L_i^j\,(\partial/\partial\mathbf{x}, \partial/\partial t) = (c_1^2 - c_2^2)\,\partial^2/\partial x_i\partial x_j + \delta_i^j\,(c_2^2\Delta - \partial^2/\partial t^2), \quad c_1 = \sqrt{(\lambda + 2\mu)/\rho}, \quad c_2 = \sqrt{\mu/\rho}$$

where c_1, c_2 are the velocities of dilatational waves and shear waves, $\delta_i^j\ (\delta_{ij})$ is the Kronecker delta and G_i are the Cartesian components of the body force.

It is well-known /2/ that system (1.3) is strictly hyperbolic. The determinant of its characteristic matrix

$$\{L_i^j\,(i\xi, i\omega)\} = \{(c_2^2 - c_1^2)\,\xi_i\xi_j - \delta_i^j\,(c_2^2\,|\,\xi\,|^2 - \omega^2)\}$$
$$(\xi = (\xi_1, \ldots, \xi_N), \quad |\,\xi\,| = \sqrt{\xi_i\xi_i})$$

has $2N$ real roots counting multiplicities (when $N = 3$ there are six roots $\pm c_1, \pm c_2, \pm c_2$; when $N = 2$ there are four: $\pm c_1, \pm c_2$). The matrix $\{-L_L^j\,(i\xi, 0)\}$ is positive definite when $|\,\xi\,| \neq 0$.

Hyperbolic systems are known to have discontinuous solutions. The surface of discontinuity is a characteristic surface of system (1.3) and it moves in the space R_N with time t. Let F_t be such a surface in R_N and F "the same" surface but in $R_{N+1} = R_N \times t, -\infty < t < \infty$, where it is stationary; let $F(\mathbf{x}, t) = 0$ be the equation of the surface, $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_N)$ a unit vector along the normal to F in R_{N+1}:

$$\nu_j = F_{,j}/\,\|\,\mathrm{grad}\,F\,\|, \quad \|\,\mathrm{grad}\,F\,\| = \sqrt{F_{,j}F_{,j}}, \quad j = 1, \ldots, N + 1, \quad x_{N+1} = t \tag{1.4}$$

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and $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ the normal to F_t in R_N:

$$n_j = F_{,j} / \| \operatorname{grad} F_t \|, \quad \| \operatorname{grad} F_t \| = \sqrt{F_{,j} F_{,j}}, \quad j = 1, \ldots, N \tag{1.5}$$

The surface F_t propagates in R_N at a velocity

$$c = -F_{,t} / \| \operatorname{grad} F_t \| \tag{1.6}$$

and its equation is

$$\det \{(c_1^2 - c_2^2) \nu_i \nu_j + \delta_{ij} (c_2^2 \sum_{k=1}^{N} \nu_k^2 - \nu_t^2)\} = 0, \quad \nu_t = \nu_{N+1} \tag{1.7}$$

Since the system is hyperbolic, Eq.(1.7) has roots

$$\nu_t = \pm c_l \left(\sum_{k=1}^{3} \nu_k^2\right)^{1/2}, \quad l = 1, 2 \tag{1.8}$$

Any characteristic surface (wave front) satisfies one of these equations; by (1.6) it moves at velocity c_L.

The condition that the displacements be continuous across the wave front, which is necessary to maintain the continuity of the medium,

$$[u_i]_{F_t} = 0 \tag{1.9}$$

implies the well-known compatibility conditions for the solutions on the moving fronts /2/:

$$[c u_{i,j} + n_j u_{i,t}]_{F_t} = 0 \tag{1.10}$$

(the continuity of the tangential derivatives of $\mathbf{u}$ on F_t). Here $[f]_{F_t}$ denotes the jump of f across F_t:

$$[f]_{F_t} = \lim_{\varepsilon \to \infty} (f(\mathbf{x} + \varepsilon \mathbf{n}, t) - f(\mathbf{x} - \varepsilon \mathbf{n}, t))$$

for $\mathbf{x} \in F_t$, $\varepsilon > 0$; $[\mathbf{n} f]_{F_t} \triangleq \mathbf{n} [f]_{F_t}$.

In addition, Eqs.(1.3) imply dynamic compatibility conditions for the solutions on the fronts /2/:

$$[\sigma_{ij} n_j + \rho c u_{i,t}]_{F_t} = 0 \tag{1.11}$$

which are equivalent to the law of conservation of momentum in the vicinity of the front.

In order to incorporate singular body forces in the equations of motion and construct fundamental solutions, the equations must be written in the space of generalized functions taking conditions (1.9)-(1.11) into account. The fundamental space $D_N(R_{N+1})$ will be the space of compactly-supported ifinitely differentiable vector functions $\varphi(\mathbf{x}, t) = \{\varphi_1(\mathbf{x}, t), \ldots, \varphi_N(\mathbf{x}, t)\}$ defined on R_{N+1} $((\mathbf{x}, t) \in R_{N+1})$. The corresponding dual space $D_N'(R_{N+1})$ is the space of generalized vector functions $\mathbf{f}^*(\mathbf{x}, t) = \{f_1^*(\mathbf{x}, t), \ldots, f_N^*(\mathbf{x}, t)\}$. Throughout, instead of "vector function" we shall always say just "function". Convergence is defined by analogy with convergence in $D(R_N) = D_1(R_N)$, $D'(R_N) = D_1'(R_N)$ /3/.

Let $\mathbf{u}(\mathbf{x}, t)$ be any classical solution of Eq.(1.3) which is continuous and twice piecewise differentiable everywhere except possibly at the surface (1.7), where conditions (1.9)-(1.11) are satisfied. Corresponding to $\mathbf{u}(\mathbf{x}, t)$ we have a generalized function $\mathbf{u}^*(\mathbf{x}, t)$:

$$(\mathbf{u}^*, \varphi) = \int_{R_{N+1}} u_i(\mathbf{x}, t) \varphi_i(\mathbf{x}, t) \, dv, \quad \forall \varphi \in D_N(R_{N+1}) \tag{1.12}$$

where the integral is evaluated over the space R_{N+1}, or, more precisely, over part of it, since $\varphi(\mathbf{x}, t)$ has bounded support. The generalized stress and strain tensors σ_{ij}^*, ε_{ij}^* are defined by (1.1), but now in the generalized sense, i.e., the generalized derivatives of $\mathbf{u}^*$ are defined by the formula /3/

$$(\mathbf{u}_{,j}^*, \varphi) = -(\mathbf{u}^*, \varphi_{,j}), \quad j = 1, \ldots, N + 1 \tag{1.13}$$

The characteristic function of the set $F_+ = \{(\mathbf{x}, t) \colon F(\mathbf{x}, t) > 0\}$ is defined as

$$H_F^+(\mathbf{x}, t) = \begin{cases} 1, & F(\mathbf{x}, t) > 0 \\ 1/2, & F(\mathbf{x}, t) = 0 \\ 0, & F(\mathbf{x}, t) < 0 \end{cases} \tag{1.14}$$

the definitions of F_- and H_F^-: $H_F^+ + H_F^- = 1$ are similar.

It is well-known /3/ that

$$H_{F,j}^+ = \nu_j \delta_F (\mathbf{x}, t), \quad H_{F,j}^- = -\nu_j \delta_F (\mathbf{x}, t) \tag{1.15}$$

$$u_{i,j}^* = u_{i,j} + [u_i]_F \nu_j \delta_F (\mathbf{x}, t). \tag{1.16}$$

Here $\nu\delta_F (\mathbf{x}, t)$ is a simple layer on F:

$$(\nu\delta_F, \varphi) = \int_F \nu_j (\mathbf{x}, t) \varphi_j (\mathbf{x}, t)\, ds \tag{1.17}$$

(the integral is evaluated over F). The first term on the right of (1.16) is the classical derivative of u_i. It follows from (1.15) and (1.16) that

$$\begin{gathered}\varepsilon_{ij}^* = \varepsilon_{ij} + 0{,}5\, [u_i\nu_j + u_j\nu_i]_F \delta_F \\ \sigma_{ij}^* = \sigma_{ij} + [\lambda u_k \nu_k \delta_{ij} + \mu (u_i\nu_j + u_j\nu_i)]_F \delta_F \\ \sigma_{ij,k}^* = \sigma_{ij,k} + [\sigma_{ij}\nu_k]_F \delta_F + \frac{\partial}{\partial x_k} \{[\lambda u_l \nu_l \delta_{ij} + \mu (u_i\nu_j + u_j\nu_i)]_F \delta_F\} \\ u_{i,tt}^* = u_{i,tt} + [\nu_t u_{i,t}]_F \delta_F + \frac{\partial}{\partial t} \{[u_i \nu_t]_F \delta_F\}\end{gathered} \tag{1.18}$$

Hence it follows that

$$\begin{gathered}\sigma_{ij,j}^* - \rho u_{i,tt}^* + \rho G_i = \sigma_{ij,j} - \rho u_{i,tt} + \rho G_i + [\sigma_{ij}\nu_j - \rho \nu_t u_{i,t}]_F \delta_F + \frac{\partial}{\partial x_j} \times \\ \{[\lambda u_k \nu_k \delta_{ij} + \mu (u_i\nu_j + u_j\nu_i)]_F \delta_F\} - \rho \frac{\partial}{\partial t} \{[u_i \nu_t]_F \delta_F\}\end{gathered} \tag{1.19}$$

By virtue of (1.2), (1.11), (1.9) and (1.6), the right-hand side of (1.19) vanishes. Consequently, u^* satisfies the same equations, but now in the generalized sense.

2. Generalized Kirchhoff-Somigliana formulae for the unsteady-state problem. Let S be the surface bounding the domain S^- of definition in R_N of a classical solution $\mathbf{u} (\mathbf{x}, t)$ of Eqs.(1.3), and $\mathbf{n}$ the unit vector of the outward normal to S, which is continuous on S. Consider the generalized function $\mathbf{u}^* (\mathbf{x}, t)$ extended by defining it as zero in the complement $S^+ = R_N \setminus (S + S^-)$: $\mathbf{u}^* (\mathbf{x}, t) = \mathbf{u} (\mathbf{x}, t) H_S^- (\mathbf{x}) H (t)$, where $H (t)$ is the Heaviside function, $H_S^- (\mathbf{x})$ the characteristic function of S^- in R_N. Both S and $t = 0$ are surfaces of discontinuity for this function. Differentiating $\mathbf{u}^*$ as in Sect.1 taking into account the equality $H^{\cdot} (t) = \delta (t)$, we obtain

$$\begin{gathered}\rho L_i^j (\partial/\partial \mathbf{x}, \partial/\partial t)\, u_j^* = -\sigma_{ij} n_j \delta_S (\mathbf{x}) H (t) - \frac{\partial}{\partial x_j} \{(\lambda u_k n_k \delta_{ij} + \mu (u_i n_j + u_j n_i)) \times \\ \delta_S (\mathbf{x}) H (t)\} - u_{i0}^{\cdot} H_S^- (\mathbf{x}) \delta (t) - u_{i0} H_S^- (\mathbf{x}) \delta^{\cdot} (t) - G_i^* \\ G_i^* = G_i H_S^- (\mathbf{x}) H (t)\end{gathered} \tag{2.1}$$

Here $H_S^- (\mathbf{x}) \delta (t)$, $H_S^- (\mathbf{x}) \delta^{\cdot} (t)$ are simple and double layers on the base of the cylinder $S^- \times T$ $(T = \{t\colon t \geqslant 0\})$ and $\delta_S (\mathbf{x}) H (t)$ is a simple layer on its lateral surface. Since $\mathbf{u}^* = 0$ outside S^- and at $t < 0$, the jumps in Eq.(2.1) are replaced by the appropriate expressions on S; $u_{i0} = u_i (\mathbf{x}, 0)$, $u_{i0}^{\cdot} = \partial u_i (\mathbf{x}, 0)/\partial t$.

Let $U_{ik}^* (\mathbf{x}, t)$ be a fundamental solution (Green's tensor) of Eq.(1.3) for a body force $G_i^* = \delta_{ik} \delta (\mathbf{x}, t)$:

$$L_i^j (\partial/\partial \mathbf{x}, \partial/\partial t)\, U_{jk}^* + \delta_{ik} \delta (\mathbf{x}, t) = 0 \tag{2.2}$$

Put $p_k = \sigma_{kj} n_j$ for $\mathbf{x} \in S$. We shall use the property of the fundamental solutions: for any $G^* \in D_N (R_{N+1})$ the corresponding solution of (1.3) is a convolution with respect to $(\mathbf{x}, t)$:

$$u_j^* = U_{jk}^* * G_k^* \tag{2.3}$$

if it exists. In view of the differentiation property of convolutions, it follows from (2.1) and (2.3) that

$$\begin{gathered}\rho u_i^* = U_{ik}^* * p_k \delta_S (\mathbf{x}) H (t) + (\lambda u_l n_l \delta_{kj} + \\ \mu (u_k n_j + u_j n_k))\, \delta_S (\mathbf{x}) H (t) * U_{ik,j}^* + \\ u_{k0}^{\cdot} H_S^- (\mathbf{x})\, \mathbf{x}^* U_{ik}^* + u_{k0} H_S^- (\mathbf{x})\, \mathbf{x}^* U_{ik,t}^* + U_{ik}^* * G_k^*\end{gathered} \tag{2.4}$$

The symbol $\mathbf{x}^*$ indicates that the convolution is evaluated with respect to $\mathbf{x}$ only.

This formula may be written in integral form, changing the notation for the dummy indices over which the summation of u_j is performed:

$$\rho u_i(\mathbf{x}, t) H(t) = \int_0^t d\tau \int_S \{U_{ik}^*(\mathbf{x}-\mathbf{y}, \tau) p_k(\mathbf{y}, t-\tau) + u_k(\mathbf{y}, t-\tau) \times \tag{2.5}$$

$$(\lambda U_{il,l}^*(\mathbf{x}-\mathbf{y}, \tau) n_k(\mathbf{y}) + \mu n_j(\mathbf{y}) (U_{ik,j}^*(\mathbf{x}-\mathbf{y}, \tau) + U_{ij,k}^*(\mathbf{x}-\mathbf{y}, \tau)))\} ds(\mathbf{y}) -$$

$$\int_{S^-} (\dot{u}_{k0}(\mathbf{y}) U_{ik}^*(\mathbf{x}-\mathbf{y}, t) + u_{k0}(\mathbf{y}) U_{ik,t}^*(\mathbf{x}-\mathbf{y}, t)) dv(\mathbf{y}) +$$

$$\int_0^t d\tau \int_{S^-} U_{ik}^*(\mathbf{x}-\mathbf{y}, \tau) G_k^*(\mathbf{y}, t-\tau) dv(\mathbf{y})$$

$$U_{ik,j}^*(\mathbf{x}-\mathbf{y}, \tau) = \partial U_{ik}^*(\mathbf{x}-\mathbf{y}, \tau)/\partial x_j$$

Defining the tensors

$$U_{ik}(\mathbf{x}, \mathbf{y}, t) = U_{ik}^*(\mathbf{x}-\mathbf{y}, t) \tag{2.6}$$

$$S_{ijk}(\mathbf{x}, \mathbf{y}, t) = \lambda\delta_{ij} U_{lk,l} + \mu (U_{ik,j} + U_{jk,i})$$

$$\Gamma_{ik}(\mathbf{x}, \mathbf{y}, t, \mathbf{n}) = S_{ijk} n_j, \quad T_{ik}(\mathbf{x}, \mathbf{y}, t, \mathbf{n}) = \Gamma_{ki}(\mathbf{y}, \mathbf{x}, t, \mathbf{n})$$

we can write formulae (2.5) in the traditional form, using the properties of Green's tensor:

$$U_{ij}^*(\mathbf{x}-\mathbf{y}, t) = U_{ji}^*(\mathbf{x}-\mathbf{y}, t) = U_{ij}^*(\mathbf{y}-\mathbf{x}, t) \tag{2.7}$$

whose properties follow from the isotropy of the medium, which implies that the equations of motion (2.2) must be invariant with respect to the group of orthogonal transformations, which of course includes the reflections. Thus, using (2.6) and (2.7), we obtain a formula of the same type as the Somigliana identity of static elasticity theory /1, 4/:

$$\rho u_i(\mathbf{x}, t) H_{S^-}(\mathbf{x}) H(t) = \int_0^t d\tau \int_S U_{ik}(\mathbf{x}, \mathbf{y}, \tau) p_k(\mathbf{y}, t-\tau) ds(\mathbf{y}) - \tag{2.8}$$

$$\int_0^t d\tau \int_S T_{ik}(\mathbf{x}, \mathbf{y}, \tau) u_k(\mathbf{y}, t-\tau) ds(\mathbf{y}) + \int_{S^-} (\dot{u}_{k0}(\mathbf{y}) U_{ik}(\mathbf{x}, \mathbf{y}, t) +$$

$$u_{k0}(\mathbf{y}) U_{ik,t}(\mathbf{x}, \mathbf{y}, t)) dv(\mathbf{y}) + \int_0^t d\tau \int_{S^-} G_k(\mathbf{y}, t-\tau) U_{ik}(\mathbf{x}, \mathbf{y}, \tau) dv(\mathbf{y})$$

The specific form of this formula depends on the form of the tensors U_{ik}, T_{ik}, $U_{ik,t}$. As all or some of these tensors are usually expressed in terms of singular generalized functions, formula (2.8) as it stands is formal, though it is frequently encountered in the literature /1/. A preferable notation is (2.4), in which the differentiation operation can be eliminated by using the properties of convolutions:

$$\rho u_i^* = p_k \delta_S(\mathbf{x}) H(t) * U_{ik}^* + \frac{\partial}{\partial x_j} \{(\lambda u_l n_l \delta_{kj} + \mu (u_k n_j + u_j n_k)) \delta_S(\mathbf{x}) H(t) * U_{ik}^*\} + \tag{2.9}$$

$$\dot{u}_{k0} H_{S^-}(\mathbf{x}) \, \mathbf{x}^* U_{ik}^* + \frac{\partial}{\partial t} \{u_{k0} H_{S^-}(\mathbf{x}) \, \mathbf{x}^* U_{ik}^*\} + G_k^* * U_{ik}^*$$

Here, if U_{ik}^* is a regular generalized function, all the convolutions can be expressed as integrals, with the differentiation applied outside the integral signs. The resulting equations may thus be investigatedin the context of continuous piecewise differentiable functions

3. ***A generalized Gauss formula for dynamic problems.*** We shall now show that the tensor $T_{ik}(\mathbf{x}, \mathbf{y}, t, \mathbf{n})$ is a fundamental solution of Eqs.(1.3). Fixing y in (2.6), we obtain

$$-T_{ik}(\mathbf{x}, \mathbf{y}, t, \mathbf{n}) = K_l^k(\partial/\partial \mathbf{x}, \mathbf{n}) U_{il}(\mathbf{x}, \mathbf{y}, t) = \tag{3.1}$$

$$\{\lambda n_k \partial/\partial x_l + \mu n_j (\delta_{lk} \partial/\partial x_j + \delta_{lk} \partial/\partial x_k)\} U_{il}(\mathbf{x}, \mathbf{y}, t)$$

$$L_j^i(\partial/\partial \mathbf{x}, \partial/\partial t) T_{ik}(\mathbf{x}, \mathbf{y}, t, \mathbf{n}) = K_j^k(\partial/\partial \mathbf{x}, \mathbf{n}) \delta(\mathbf{x}-\mathbf{y}, t)$$

The last equation follows from (2.5), and we rewrite it as

$$L_j^i(\partial/\partial \mathbf{x}, \partial/\partial t) T_{ik}(\mathbf{x}, \mathbf{y}, t, \mathbf{n}) = \lambda n_k \partial\delta/\partial x_j + \mu n_l (\delta_{jk} \partial\delta/\partial x_l + \delta_{jl} \partial\delta/\partial x_k) \tag{3.2}$$

$$\left(\partial\delta/\partial x_l = \delta(t) \, \delta^{\cdot}(x_l - y_l) \prod_{i \neq l} \delta(x_i - y_i)\right)$$

By Eqs.(1.2) with $y=0$,

$$S_{ijk,j}-\rho U_{ik,tt}+\rho\delta_{ik}\delta(\mathbf{x},t)=0. \tag{3.3}$$

Convolving (3.3) with $H_S^-(\mathbf{x})H(t)$ and using (1.15), we obtain

$$S_{ijk,j}*H_S^-(\mathbf{x})H(t)=-\int\limits_{ijk}*n_j\delta_S(\mathbf{x})H(t)= \tag{3.4}$$

$$-\rho\delta_{ik}H_S^-(\mathbf{x})H(t)+\rho\frac{\partial^2}{\partial t^2}\{U_{ik}*H_S^-(\mathbf{x})H(t)\}.$$

We now use (2.6) and recast (3.4) in integral form:

$$\int\limits_0^t d\tau\int\limits_{S^-}T_{ik}(\mathbf{y},\mathbf{x},\tau,\mathbf{n}(\mathbf{y}))\,ds(\mathbf{y})=\rho\delta_{ik}H_S^-(\mathbf{x})H(t)-\rho\frac{\partial}{\partial t}\int\limits_{S^-}U_{ik}(\mathbf{x},\mathbf{y},t)\,dv(\mathbf{y}) \tag{3.5}$$

Unlike the Gauss formula of static elasticity theory /4/, this equation involves a second term on the right, representing time-dependence.

4. The tensors U_{ik}^* *and* T_{ik}^*. In two dimensions $(N=2)$ the tensor U_{ik}^* was constructed in /5/, but the development there involves an error (see below), because of which the resulting formula for U_{ik}^* is incorrect. The tensor U_{ik}^* in three dimensions was worked out in /1/. The simple approach adopted here will be different.

Evaluating the generalized Fourier transform of (2.2) and solving the equations thus obtained, we obtain the Fourier transform of Green's tensor:

$$F[U_{ij}^*(\mathbf{x},t)]=\frac{\delta_{ij}}{c_2^2|\xi|^2-\omega^2}+\frac{\xi_i\xi_j}{\omega^2}\left(\frac{c_1^2}{c_1^2|\xi|^2-\omega^2}-\frac{c_2^2}{c_2^2|\xi|^2-\omega^2}\right) \tag{4.1}$$

where $(\xi_1,\ldots,\xi_N,\omega)$ are the Fourier variables corresponding to $(x_1,\ldots,x_N,t)$ $|\xi|=\sqrt{\xi_i\xi_i}$. The generalized Fourier transform is defined by

$$(F[\mathbf{f}^*(\mathbf{x},t)],\ F[\varphi(\mathbf{x},t)])=(2\pi)^{N+1}(\mathbf{f}^*(\mathbf{x},t),\ \varphi(\mathbf{x},t)) \tag{4.2}$$

$$(F[\varphi(\mathbf{x},t)]=\int\limits_{R_{N+1}}\varphi(\mathbf{x},t)\exp(i(\xi\mathbf{x})+i\omega t)\,dv$$

for any $\varphi\in D_N(R_{N+1}))$.

It is obvious that the function

$$\Phi_0(\xi,\omega,c)=(c^2|\xi|^2-\omega^2)^{-1} \tag{4.3}$$

is the Fourier transform of Green's function of the D'Alembert wave equation

$$(\partial^2/\partial t^2-c^2\Delta)\,\Phi_0(\mathbf{x},t,c)=\delta(\mathbf{x},t) \tag{4.4}$$

whose solutions are readily available for any N /3/. The functions

$$\overline{\Phi}_1=-\overline{\Phi}_0/(i\omega),\quad \overline{\Phi}_2=-\overline{\Phi}_1/(i\omega)=\overline{\Phi}_0/(i\omega)^2$$

are the Fourier transforms of the convolutions

$$\overline{\Phi}_1=F[\Phi_0*H(t)\,\delta(\mathbf{x})],\quad \overline{\Phi}_2=F[\Phi_1*H(t)\,\delta(\mathbf{x})]$$

if the regularization of the function $1/(i\omega)$ is taken to be $1/(i(\omega+i0))$, as $\Phi_j=0$ for $t<0$. Consequently,

$$\Phi_1=\Phi_0*H(t)\,\delta(\mathbf{x})=H(t)\int\limits_0^t\Phi_0(\mathbf{x},\tau,c)\,d\tau \tag{4.5}$$

$$\Phi_2=\Phi_1*H(t)\,\delta(\mathbf{x})=H(t)\int\limits_0^t\Phi_1(\mathbf{x},\tau,c)\,d\tau$$

Since

$$F[\partial\mathbf{f}^*/\partial x_k]=-i\xi_kF[\mathbf{f}^*].$$

we deduce from (4.1)

$$U_{ik}^*(\mathbf{x},t)=\Phi_0(\mathbf{x},t,c_2)\,\delta_{ik}+\partial^2(c_1^2\Phi_2(\mathbf{x},t,c_1)-c_2^2\Phi_2(\mathbf{x},t,c_2))\,\partial^2/\partial x_i\partial x_k \tag{4.6}$$

and from (3.1), in view of (4.6), that

$$T_{ik}^* (\mathbf{x}, t, \mathbf{n}) = \lambda n_k \Phi_{01,i} + \mu ((n_i \Phi_{02,k} + \delta_{ik} \partial\Phi_{02}/\partial \mathbf{n}) + 2\partial (c_1^2 \Phi_{21,ik} - c_2^2 \Phi_{22,ik}))/\partial \mathbf{n} \tag{4.7}$$
$$\Phi_{kj} = \Phi_k (\mathbf{x}, t, c_j), \quad \partial/\partial\mathbf{n} = n_j \partial/\partial x_j$$

Two-dimensional deformation. For $N = 2$ we have /3/

$$\Phi_0 (\mathbf{x}, t, c) = \frac{H(ct - r)}{2\pi c \sqrt{c^2 t^2 - r^2}}, \quad r = \sqrt{x_1^2 + x_2^2} \tag{4.8}$$

Implementing the integration in (4.5), we find that

$$\Phi_k (x, t, c) = H (ct - r) f_k (r, t, c), \quad k = 0, 1, 2 \tag{4.9}$$
$$f_1 (r, t, c) = \frac{1}{2\pi c^2} \ln \frac{ct + \sqrt{c^2 t^2 - r^2}}{r}$$
$$f_2 (r, t, c) = \frac{1}{2\pi c^3} \left(ct \ln \frac{ct + \sqrt{c^2 t^2 - r^2}}{r} - \sqrt{c^2 t^2 - r^2} \right)$$

Substituting (4.9) into (4.6), we obtain U_{ik}^* in the two-dimensional case*: (*The evaluation of the functions analogous to our f_1, f_2, in /5/ involved an error; in particular, r was omitted from the denominators of the expressions under the logarithm sign in formulae (4.3.155)-(4.3.155").

$$U_{ik}^* (x, t) = \frac{1}{2\pi} \left\{ \frac{t^2}{r^2} (2r_{,i} r_{,k} - \delta_{ik}) \left(\frac{c_1 H (c_1 t - r)}{\sqrt{c_1^2 t^2 - r^2}} - \frac{c_2 H (c_2 t - r)}{\sqrt{c_2^2 t^2 - r^2}} \right) + \frac{H (c_1 t - r)}{c_1 \sqrt{c_1^2 t^2 - r^2}} (\delta_{ik} - r_{,i} r_{,k}) + \frac{H (c_2 t - r)}{c_2 \sqrt{c_2^2 t^2 - r^2}} r_{,i} r_{,k} \right\}, \quad r_{,k} = \frac{\partial r}{\partial x_k} = \frac{x_k}{r} \tag{4.10}$$

It follows from (4.10) that U_{ik}^* is a regular generalized function, with integrable (in R_3) singularities of order $(c_j^2 t^2 - r^2)^{-1/2}$ on two fronts $K^j = \{(\mathbf{x}, t) \in R_3\colon r = c_j t\}$. The moving fronts $K_t^j = \{\mathbf{x} \in R_2\colon r = c_j t\}$, which are circles of radius $c_j t$, expand in R_2 at a rate c_j. Ahead of the front K_t^1 $U_{ik}^* = 0$.

At $r = 0$, $t \neq 0$ the tensor U_{ik}^* has a removable singularity, as shown by the asymptotic formula

$$\frac{t^2}{r^2} \left(\frac{c_1 H (c_1 t - r)}{\sqrt{c_1^2 t^2 - r^2}} - \frac{c_2 H (c_2 t - r)}{\sqrt{c_2^2 t^2 - r^2}} \right) \sim \frac{c_2^{-2} - c_1^{-2}}{2t}, \quad r \to 0 \tag{4.11}$$

It can be shown that

$$\frac{\partial H (ct - r)}{\partial x_j} = -\frac{x_j}{r} \delta (ct - r) H (t) = -\frac{x_j}{ct} \delta (ct - r) H (t) \tag{4.12}$$

Here the right-hand side corresponds to a simple layer on the surface of a cone $K = \{(\mathbf{x}, t)\colon r = ct, \ t > 0\}$:

$$\left(\frac{x_j}{r} \delta (ct - r) H (t), \quad \varphi_j (\mathbf{x}, t) \right) = \int_0^\infty dt \int_{r=ct} \frac{x_j}{ct} \varphi_j (\mathbf{x}, t)\, ds, \quad \varphi \in D_N (R_{N+1}) \tag{4.13}$$

where the inner integral is evaluated along a circle of radius ct. It follows that the tensor T_{ik}^* is a singular generalized function, whose precise form may be determined by using (4.7).

Three-dimensional deformation. When $N = 3$ /3/ we have

$$\Phi_0 (\mathbf{x}, t, c) = \frac{\delta (ct - r) H (t)}{4\pi c r}, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2} \tag{4.14}$$

Formulae (4.5) imply

$$\Phi_1 (\mathbf{x}, t, c) = \frac{H (ct - r)}{4\pi c^2 r}, \quad \Phi_2 (\mathbf{x}, t, c) = \frac{H (ct - r) (t - r/c)}{4\pi c^2 r} \tag{4.15}$$

Substituting into (4.6), we find that

$$U_{ik}^* (\mathbf{x}, t) = t (4\pi r^2)^{-1} \{\delta (c_2 t - r) H (t) (\delta_{ik} - r_{,i} r_{,k}) + tr^{-1} (\delta_{ik} - 3r_{,i} r_{,k}) (H (c_2 t - r) - H (c_1 t - r)) + \delta (c_1 t - r) H (t) r_{,i} r_{,k}\} \quad i, k = 1, 2, 3 \tag{4.16}$$

Formula (4.16) was first derived by Stokes /1/, via direct inversion of the Fourier-Laplace transform of U_{1k}^*.

5. *Integral analogues of formula (2.8) for* $N = 2$. We will first consider the problem with vanishing initial data and body forces: $u_{i0} = 0$, $u_{i0}^{\cdot} = 0$, $G_i = 0$.

When $N = 3$ formula (2.8) cannot be used, since V_{ij}^* is a singular generalized function with simple layers on the surfaces of the cones K_t^1, K_t^2. Its regular part, which includes $H(c_j t - r)$, is non-zero only between the fronts. When $N = 2$ the tensor $T_{ij}^* (\mathbf{x}, \mathbf{y}, t, \mathbf{n})$ involves non-integrable singularities of the form $(r - c_j t)^{-3/2}$, $r = \| \mathbf{x} - \mathbf{y} \|$, so that here too formula (2.8) cannot be used to determine $u_j (\mathbf{x}, t)$. We shall use formula (2.9) to construct an integral analogue in the case $N = 2$.

Express V_{ik}^* as $U_{ik}^* = U_{ik1} + U_{ik2}$, where U_{ikj}^* are the terms which depend on c_j in (4.10). U_{ik1}^* describes a volumetric deformation and T_{ik2}^* a shear deformation. Similar decompositions hold for the tensors T_{ik}^*, U_{ik}, T_{ik}. Put

$$W_{ikj} (\mathbf{x}, \mathbf{y}, t) = \int_{r/c_j}^{t} U_{ikj} (\mathbf{x}, \mathbf{y}, \tau)\, d\tau \tag{5.1}$$

$$H_{ikj} (\mathbf{x}, \mathbf{y}, t, \mathbf{n}) = \lambda n_k \frac{\partial W_{imj}}{\partial y_m} + \mu n_m \left(\frac{\partial W_{ikj}}{\partial y_m} + \frac{\partial W_{imj}}{\partial y_k}\right), \quad H_{ik} = H_{ik1} + H_{ik2}$$

Clearly,

$$W_{ikj} (\mathbf{x}, \mathbf{y}, r/c_j) = 0 \tag{5.2}$$

and it follows from (3.5) that

$$\int_S H_{ik} (\mathbf{y}, \mathbf{x}, t, \mathbf{n} (\mathbf{y}))\, ds (\mathbf{y}) = \rho \delta_{ik} H_S^- (\mathbf{x}) H (t) - \rho \frac{\partial}{\partial t} \int_{S^-} U_{ik} (\mathbf{x}, \mathbf{y}, t)\, dv (\mathbf{y}) \tag{5.3}$$

Using (4.5), (4.6) and (4.10), we obtain

$$W_{ik} (\mathbf{x}, \mathbf{y}, t) = W_{ik1} + W_{ik2} = \frac{H (c_2 t - r)}{2\pi c_2^2} \delta_{ik} \ln \frac{c_2 t + \sqrt{c_2^2 t^2 - r^2}}{r} - \sum_{j=1}^{2} (-1)^j \frac{H (c_j t - r)}{2\pi c_j^2} \times \left(\delta_{ik} \ln \frac{c_j t + \sqrt{c_j^2 t^2 - r^2}}{r} + \frac{2r_{,i} r_{,k} - \delta_{ik}}{r^2} c_j t \sqrt{c_j^2 t^2 - r^2}\right) \tag{5.4}$$

Since as $r \to 0$

$$tr^{-2} (c_1^{-1} \sqrt{c_1^2 t^2 - r^2} - c_2^{-1} \sqrt{c_2^2 t^2 - r^2}) \sim {}^1/_2 (c_2^{-2} - c_1^{-2})$$

it follows that the singularity of the tensor W_{ik} at $r = 0$ is only logarithmic. Accordingly, H_{ik} has a singularity of type $1/r$.

Consider formula (2.9). U_{ik}^* is a regular generalized function, and we may therefore write all the convolutions as integrals:

$$\rho u_i^* (\mathbf{x}, t) = \int_0^t d\tau \int_S U_{ik} (\mathbf{x}, \mathbf{y}, \tau)\, p_k (\mathbf{y}, t - \tau)\, ds (\mathbf{y}) + \frac{\partial}{\partial x_j} \int_0^t d\tau \int_S \{\lambda U_{ij} (\mathbf{x}, \mathbf{y}, \tau)\, n_m (\mathbf{y}) + \mu n_j (\mathbf{y}) U_{im} (\mathbf{x}, \mathbf{y}, \tau)\} u_m (\mathbf{y}, t - \tau)\, ds (y) + \frac{\partial}{\partial x_m} \int_0^t d\tau \int_S \mu U_{ik} (\mathbf{x}, \mathbf{y}, \tau)\, n_k (\mathbf{y}) \times u_m (\mathbf{y}, t - \tau)\, ds (\mathbf{y}) \tag{5.5}$$

where all the integrals exist; they may be written differently, e.g.,

$$\int_0^t d\tau \int_S U_{ik} (\mathbf{x}, \mathbf{y}, \tau)\, p_k (\mathbf{y}, t - \tau)\, ds (\mathbf{y}) = \tag{5.6}$$

$$\sum_{j=1}^{2}\int_{S_t^j} ds(\mathbf{y})\int_{r/c_j}^{t} U_{ikj}(\mathbf{x},\mathbf{y},\tau)\,p_k(\mathbf{y},t-\tau)\,d\tau$$

$$(S_t^j = \{\mathbf{y}\in S:\ \|\mathbf{x}-\mathbf{y}\| < c_j t\})$$

In order to differentiate under the integral sign, we introduce regularization at the front:

$$\rho u_L^*(\mathbf{x},t) = \sum_{k=1}^{2}\int_0^t dt\int_{S_\tau^k} U_{ijk}(\mathbf{x},\mathbf{y},\tau)\,p_j(\mathbf{y},t-\tau)\,ds(\mathbf{y}) + \tag{5.7}$$

$$\frac{\partial}{\partial x_j}\int_0^t d\tau\int_{S_\tau^k}\left(u_m(\mathbf{y},t-\tau)-u_m\left(\mathbf{y},t-\frac{r}{c_k}\right)\right)\{\lambda U_{ijk}(\mathbf{x},\mathbf{y},\tau)\,n_m(\mathbf{y}) +$$

$$\mu U_{imk}(\mathbf{x},\mathbf{y},\tau)\,n_j(\mathbf{y})\}\,ds(\mathbf{y}) + \frac{\partial}{\partial x_m}\int_0^t d\tau\int_{S_\tau^k}\mu U_{ijk}(\mathbf{x},\mathbf{y},\tau)\,n_j(\mathbf{y})\left(u_m(\mathbf{y},t-\tau)-\right.$$

$$\left.u_m\left(\mathbf{y},t-\frac{r}{c_k}\right)\right)ds(\mathbf{y}) + \frac{\partial}{\partial x_j}\int_{S_t^k} u_m\left(\mathbf{y},t-\frac{r}{C_k}\right)(\lambda n_m(\mathbf{y})\,W_{ijk}(\mathbf{x},\mathbf{y},t) +$$

$$\mu n_j(\mathbf{y})\,W_{imk}(\mathbf{x},\mathbf{y},t))\,ds(\mathbf{y}) + \frac{\partial}{\partial x_m}\int_{S_t^k} u_m\left(\mathbf{y},t-\frac{r}{c_k}\right)\mu n_j(\mathbf{y})\,W_{ijk}(\mathbf{x},\mathbf{y},t)\,ds(\mathbf{y})$$

The integrands in the second and third integrals have removable singularities at the fronts $r = c_k\tau$, thanks to the equality

$$\lim_{\tau\to\frac{r}{c_k}+0}\frac{u_m(\mathbf{y},t-\tau)-u_m(\mathbf{y},t-r/c_k)}{\sqrt{c_k^2\tau^2-r^2}} = \tag{5.8}$$

$$-\frac{1}{c_k}\frac{\partial u(\mathbf{y},t-r/c_k-0)}{\partial\tau}\lim_{\tau\to\frac{r}{c_k}+0}\sqrt{\frac{c_k\tau-r}{c_k\tau+r}} = 0$$

which holds for any r. On the boundary of the sets S_τ^k (at $r = c_k\tau$) they vanish (this is important if $S_\tau^k \neq S$, for then the endpoints of the interval of integration depend on $\mathbf{x}$). The integrands in the fourth and fifth integrals vanish at the boundary of S_t^k because of (5.2). All the integrands are differentiable with respect to $\mathbf{x}$. Accordingly, it is legitimate to differentiate within the integral. Collecting like terms and using (2.6), (5.4), we obtain

$$\rho u_i(\mathbf{x},t)\,H_S^-(\mathbf{x})\,H(t) = \sum_{k=1}^{2}\int_0^t d\tau\int_{S_\tau^k}\left(U_{ijk}(\mathbf{x},y,\tau)\,p_j(\mathbf{y},t-\tau)-\right. \tag{5.9}$$

$$\left.T_{ijk}(\mathbf{x},\mathbf{y},\tau)\left(u_j(\mathbf{y},t-\tau)-u_j\left(\mathbf{y},t-\frac{r}{c_k}\right)\right)\right)ds(\mathbf{y}) -$$

$$\int_{S_t^k} u_j\left(\mathbf{y},t-\frac{r}{c_k}\right)H_{ijk}(\mathbf{x},\mathbf{y},t,\mathbf{n}(\mathbf{y}))\,ds(\mathbf{y})$$

The first integral exists for any $\mathbf{x}$, the second, for $\mathbf{x}\,\overline{\in}\,S$.

Note that formula (5.9) may be derived from the formal integral equality (2.8) if the integrands are regularized at the front.

Formulae (5.3) and (5.9) have been developed for generalized functions, but both sides involve regular generalized functions. It is known /3/ that they are identical as real-valued functions in the region of continuity. Thus (5.3) and (5.9) hold in the conventional sense too. To prove that they are valid on the surface of the discontinuity S, one must let $\mathbf{x}\to S$, as is normally done in static problems /4, 6/. In the case of smooth Lyapunov surfaces formula (5.9) yields singular boundary integral equations for the solution of the boundary-value problems of elasticity theory. We shall not dwell on the proof here.

Let us assume now that the initial data are not zero. Since $U'_{ik,t}$ has a singularity $(r-c_jt)^{-3/2}$, the corresponding integral in (2.8) does not exist, so that this formula is useless. We make use instead of (2.9). Split the tensor U_{ik} as given in (4.10) into two: U_{ik}^1 describes the motion between the fronts and U_{ik}^2 the motion upstream of the wave front:

$$U_{ik}(\mathbf{x},\mathbf{y},t) = \sum_{j=1}^{2} U_{ik}^{j}(r,\mathbf{e},t), \quad \mathbf{e} = (r_{,1}, r_{,2}).$$

$$U_{ik}^{1} = \frac{H(c_1t-r)H(r-c_2t)}{2\pi c_1\sqrt{c_1^2t^2-r^2}}\left\{\left(\frac{c_1t}{r}\right)^2(2r_{,i}r_{,k}-\delta_{ik})+\delta_{ik}-r_{,i}r_{,k}\right\}$$

$$U_{ik}^{2} = \frac{H(c_2t-r)}{2\pi}\left\{\left(\frac{t}{r}\right)^2(2r_{,i}r_{,k}-\delta_{ik})\left(\frac{c_1}{\sqrt{c_1^2t^2-r^2}}-\frac{c_2}{\sqrt{c_2^2t^2-r^2}}\right)+\frac{\delta_{ik}-r_{,i}r_{,k}}{c_1\sqrt{c_1^2t^2-r^2}}+\frac{r_{,i}r_{,k}}{c_2\sqrt{c_2^2t^2-r^2}}\right\}$$

We define the tensors

$$D_{ik}^{j} = \int_0^{c_jt}\frac{r}{t}U_{ik}^{j}(r,\mathbf{e},t)\,dr$$

and evaluate them using the equalities

$$\int\frac{r\,dr}{\sqrt{c^2t^2-r^2}} = -\sqrt{c^2t^2-r^2}$$

$$\int\frac{dr}{r\sqrt{c^2t^2-r^2}} = -\frac{1}{ct}\ln\frac{ct+\sqrt{c^2t^2-r^2}}{r}$$

The results are

$$D_{ik}^{1} = \frac{1}{2\pi}\left\{(2r_{,i}r_{,k}-\delta_{ik})\ln\frac{1+\sqrt{1-\gamma^2}}{\gamma}+(\delta_{ik}-r_{,i}r_{,k})\sqrt{1-\gamma^2}\right\}$$

$$D_{ik}^{2} = \frac{1}{2\pi}\left\{(\delta_{ik}-2r_{,i}r_{,k})\ln(1+\sqrt{1-\gamma^2})+\delta_{ik}(1+\sqrt{1-\gamma^2})+r_{,i}r_{,k}\sqrt{1-\gamma^2}\right\},$$

$$\gamma = c_2/c_1$$

i.e., $D_{ij}{}^{k} = D_{ij}{}^{k}(\mathbf{e})$ and these tensors are independent of t. Consider the expansions

$$I_i(\mathbf{x},t) = U_{ik}^{*} \ast u_{k0}H_S^{-}(\mathbf{x}) = \sum_{j=1}^{2}\int_{V_t^j}U_{ik}^{j}(\mathbf{x},\mathbf{y},t)\,u_{k0}^{*}(\mathbf{y})\,dv(\mathbf{y}) \tag{5.10}$$

$$(u_{k0}^{*}(\mathbf{x}) = u_{k0}(\mathbf{x})H_S^{-}(\mathbf{x}), \quad V_t^j = \{\mathbf{y} : \|\mathbf{x}-\mathbf{y}\| < c_jt\})$$

Define the vectors $\mathbf{z}_j(\mathbf{x},\mathbf{e},t) = \mathbf{x}+\mathbf{e}c_jt$. We can rewrite the last equality differently, assuming regularization of the second type at the front:

$$I_L(\mathbf{x},t) = \sum_{j=1}^{2}\int_{V_t^j}U_{ik}^{j}(\mathbf{x},\mathbf{y},t)\,(u_{k0}^{*}(\mathbf{y})-u_{k0}^{*}(\mathbf{z}_j))\,dv(\mathbf{y}) + \int_{V_t^j}U_{ik}^{j}(\mathbf{x},\mathbf{y},t)\,u_{k0}^{*}(\mathbf{z}_j)\,dv(\mathbf{y})$$

To continue, it is convenient to transform this integral to polar coordinates with origin at x and integrate with respect to r:

$$I_L(\mathbf{x},t) = \sum_{j=1}^{2}\int_{V_t^j}U_{ik}^{j}(\mathbf{x},\mathbf{y},t)(u_{k0}^{*}(\mathbf{y})-u_{k0}^{*}(\mathbf{z}_j))\,dv(\mathbf{y}) + t\int_{\|\mathbf{e}\|=1}D_{ik}^{j}(\mathbf{e})u_{k0}^{*}(\mathbf{z}_j)\,ds(\mathbf{e}) \tag{5.11}$$

$$\frac{\partial I_L}{\partial t} = \sum_{j=1}^{2}\int_{V_t^j}U_{ik,t}^{j}(\mathbf{x},\mathbf{y},t)(u_{k0}^{*}(\mathbf{y})-u_{k0}^{*}(\mathbf{z}_j))\,dv(\mathbf{y}) + \int_{\|\mathbf{e}\|=1}D_{ik}^{j}(\mathbf{e})u_{k0}^{*}(\mathbf{z}_j)\,ds(\mathbf{e})$$

We have here cancelled out like terms obtained in the differentiation of $u_{k0}(z_j)$:

$$u_{k0,t}(\mathbf{z}_j) = e_mc_ju_{k0,m}(\mathbf{z}_j)$$

(on the assumption that $u_{k0,m} = \partial u_{k0}/\partial x_m$ exists). All the integrals in (5.11) exist.

As a result, we can write formula (2.9) in the following integral form:

$$\rho u_i(\mathbf{x}, t) H_{S^-}(\mathbf{x}) H(t) = \sum_{k=1}^{2} \int_0^t d\tau \int_{S_\tau^k} \Big(U_{ijk}(\mathbf{x}, \mathbf{y}, \tau) p_i(\mathbf{y}, t-\tau) -$$

$$T_{ijk}(\mathbf{x}, \mathbf{y}, \tau) \Big(u_j(\mathbf{y}, t-\tau) - u_j\Big(\mathbf{y}, t - \frac{r}{c_k}\Big)\Big)\Big) ds(\mathbf{y}) -$$

$$\int_{S_t^k} u_j\Big(\mathbf{y}, t - \frac{r}{c_k}\Big) H_{ijk}(\mathbf{x}, \mathbf{y}, t) ds(\mathbf{y}) + \int_{V_t^k} (U_{ij}^k(\mathbf{x}, \mathbf{y}, t) \dot{u}_{j0}(\mathbf{y}) H_{S^-}(\mathbf{y}) +$$

$$U_{ij,t}^k(\mathbf{x}, \mathbf{y}, t)(u_{j0}^*(\mathbf{y}) - u_{j0}^*(\mathbf{z}_k))) dv(\mathbf{y}) + \int_{\|\mathbf{e}\|=1} D_{ij}^k(\mathbf{e}) u_{j0}^*(\mathbf{x} + \mathbf{e}c_k t) ds(\mathbf{e}) +$$

$$\int_0^t d\tau \int_{S^-} U_{ij}(\mathbf{x}, \mathbf{y}, \tau) G_j(\mathbf{y}, t-\tau) dv(\mathbf{y})$$

If $\mathbf{x} \in S$, this formula gives singular boundary integral equations for solving the boundary-value problems of unsteady-state elasticity theory with arbitrary boundary and initial conditions. Based on the approach outlined here, one can state the necessary conditions that the boundary and initial conditions must satisfy. The sufficient conditions are their continuity and the continuity of $\partial u_{k0}/\partial x_j$.

The analogue of (5.9) for the case $N = 3$ was developed by N.M. Khutoryanskii /6/.

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